J. Fluid Mech. (1996), *vol.* 323, *pp.* 317–338

Rossby wave interaction in a shear flow with critical levels

By JACQUES VANNESTE

Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7

(Received 15 December 1995)

An approach is presented for studying Rossby wave interaction in a shear flow with both regular and singular modes (i.e. those possessing a critical level). The approach relies on a truncated normal mode expansion of the equations of motion. Such an expansion remains valid in the presence of singular modes, provided that these modes are not considered individually, but that complete packets are taken into account in the truncated system. Mathematically, this means that the interaction equations need to be integrated with respect to the phase velocity (or, equivalently, the critical level position) of the singular modes.

The action of two regular modes on a packet of singular modes is treated in detail; in particular, asymptotic results are deduced for the long-term behaviour of the packet. The case of a linear shear is considered as an illustration: analytical expressions are derived for the normal modes and their pseudomomentum, and they are used to present explicit results for the evolution of the packet of singular modes.

1. Introduction

The presence of a shear flow drastically changes the properties of stable dispersive waves propagating in a fluid, and hence modifies the properties of their weakly nonlinear interactions (e.g. Craik 1985). A striking modification is the existence of positive and negative (pseudo)energy waves (e.g. Ripa 1990), which may lead to explosive resonant interaction. This mechanism, which induces a nonlinear instability of the shear flow, was first investigated in the context of layered fluid models (Cairns 1979; Craik & Adam 1979), and its relevance to continuous shear flows has been discussed recently (Becker & Grimshaw 1993; Vanneste & Vial 1994). Another modification is the existence of a continuous spectrum of singular modes – possessing a critical level – in addition to the discrete spectrum of regular modes present in a resting basic state. Although the rôle of the continuous spectrum in linear initial-value problems (e.g. Farrell 1982; Tung 1983), as well as the forcing of an isolated singular mode (e.g. Warn & Warn 1978; Ritchie 1985; see also Maslowe 1986) have been much studied, triad interactions involving singular modes seem to have received little attention. However, such interactions certainly occur in fluids, and it is of interest to develop a proper mathematical description for studying their influence. Grimshaw (1988, 1994) considered the rôle of critical levels in wave interactions, but, in his work, resonance conditions are only met on certain space–time surfaces, whereas here they are met globally. A motivation for considering triads including singular modes is provided by a result of Becker & Grimshaw (1993) which indicates that such triads are necessary for explosive interaction to exist in a continuously stratified shear flow. Similarly, they may also be invoked for (baroclinic) Rossby waves, and therefore be involved in nonlinear baroclinic instability (Romanova 1987; Meacham 1988; Vanneste 1995).

Even in the linear domain, the definition and the properties of the singular normal modes have long constituted a perplexing issue, because of the many difficulties involved in the treatment of singular differential (or integral) equations. Case (1960) clarified this issue by showing that the singular modes are required to ensure the completeness of the normal-mode basis, and that their superposition leads to regular physical fields (see also Tung 1983). Thus the free linear evolution of an initially smooth disturbance does not lead to a singularity; rather, a packet of singular modes represents a disturbance whose energy decays for large time (possibly after a transient growth stage). No extra physical effects such as viscosity or additional nonlinearity need, therefore, to be introduced within a particular critical layer. This should be distinguished from the evolution of a forced (or an unstable) mode, in which a critical layer is created and large amplitudes are attained because a fixed frequency is continuously excited (e.g. Warn & Warn 1978; Benney & Maslowe 1975; Brown & Stewartson 1979; Maslowe, Benney & Mahoney 1994). Recent work by Kamp (1991), and Balmforth & Morrison (1996) has further clarified the free problem by showing that the singular modes can be derived from a (regular) Fredholm equation. In the non-rotating case, Balmforth & Morrison (1996) have established the completeness of the normal-mode basis, as well as its equivalence to Laplace transform theory for the linearized equations. Similar results hold when $\beta \neq 0$, and they form the basis of our analysis in the nonlinear domain.

The purpose of this note is to present an approach for treating triad interactions involving singular modes. As a specific application, we investigate the influence of two interacting regular modes on the continuous spectrum. The physical system analysed is the two-dimensional flow in a β-channel – which supports the propagation of barotropic Rossby waves – but the method as well as the main qualitative results can be applied to other fluid systems. Our approach relies upon the fact that the normal-mode expansion with suitable orthogonality relations used in Vanneste & Vial (1994) and Vanneste (1995) can be used in the presence of singular modes. When truncating the system, it is however essential that the singular modes are not considered individually but that the entire spectrum corresponding to a given zonal wavenumber is taken into account. Mathematically, this means that the interaction coefficients related to singular modes must be interpreted as distributions, which make sense only under an integral. The physical idea behind this treatment is fairly simple: in unforced problems, there is no justification for isolating a particular mode belonging to the continuous spectrum from the modes in its vicinity, as their characteristics are very similar. In particular, the resonant interaction involving a singular mode cannot be dissociated from the non-resonant interaction involving the modes in its vicinity. This situation is somewhat analogous to that arising when a mode near marginal stability belongs to a triad and cannot be dissociated from the mode with which it tends to coalesce (see Romanova 1994). Like their linear evolution, the free nonlinear evolution of singular modes does not lead to any finite-time singularity in the physical fields, provided that the evolution of a complete packet is described.

The general ideas of the paper are applied to studying the interaction between two regular modes and a packet of singular modes, a member of which forms a resonant triad with the regular modes. For simplicity, the amplitude of the regular modes is taken as constant so that only their action on the singular modes needs to be considered. Starting from an unexcited continuous spectrum, and taking into account the non-resonant interactions, it is shown that the contribution of the singular modes leads to smooth velocity and vorticity fields. However, asymptotic results for time tending to infinity indicate that these fields tend to a singular structure, which is different from that of the single resonant mode. In the same limit, the contribution of the singular spectrum to the total pseudomomentum is shown to increase linearly with time. These results are illustrated in the case of a linear shear, using the analytical solutions that can be derived for both regular and singular modes: calculations for a particular triad are described, and compared with the asymptotic estimates.

The plan of the paper is as follows. The β-plane model and the normal-mode theory are reviewed in §2. Section 3 is devoted to the derivation of the interaction equations and to their interpretation for singular modes. In §4, general expressions are established for the evolution of the streamfunction and pseudomomentum of the packet forced by two interacting regular modes, and their asymptotic behaviour is discussed. In §5, analytical expressions are given for the normal modes in a linear shear, and they are used to illustrate the theoretical results of the previous sections. In the final section, extensions of the method and the implications of the results are discussed.

2. Basic equations

The system under consideration is a barotropic flow on a β-plane, which is governed by conservation of the absolute vorticity. A disturbance to a steady parallel flow $U(y)$ obeys the evolution equation

$$q_t + Uq_x + Q'\psi_x + \partial(\psi, q) = 0, \tag{2.1}$$

where ψ is the disturbance streamfunction, $q := \nabla^2\psi$ is the disturbance vorticity, $Q' := \beta - U_{yy}$ is the meridional gradient of the absolute vorticity of the basic flow, and $\partial(\psi, q) := \psi_x q_y - \psi_y q_x$ is the Jacobian operator. This equation has been rendered dimensionless, so that we may assume $U_M := \max_y U(y) = 1$ and write the boundary conditions as

$$\psi_x = 0 \quad \text{and} \quad \int \psi_{yt}\,\mathrm{d}x = 0 \quad \text{at } y = 0, 1 \tag{2.2}$$

(see e.g. Pedlosky 1987, p. 147). The dimensionless parameter β is then related to its dimensional counterpart $\tilde{\beta}$ through $\beta = \tilde{L}^2\tilde{\beta}/\tilde{U}_M$, where $\tilde{L}$ and $\tilde{U}_M$ are the unscaled channel width and maximum velocity, respectively. Taking advantage of the x-invariance of the problem, it may also be assumed that $U_m := \min_y U(y) = 0$. In the x-direction, we consider a (possibly infinite) periodic channel, so that $\int(\cdot)\mathrm{d}x := l^{-1}\int_0^l(\cdot)\mathrm{d}x$, where l is the channel period.

2.1. *Normal modes*

The linear solutions of (2.1) for zonal wavenumber k_a are found in the form of normal modes

$$\psi = \psi_a(y)\exp[\mathrm{i}k_a(x - c_a t)] + \text{c.c.}, \tag{2.3}$$

where $c_a =: \omega_a/k_a$ is the phase velocity. The meridional structure $\psi_a(y)$ satisfies the Rayleigh–Kuo eigenvalue problem

$$(U - c_a)q_a + Q'\psi_a = 0, \quad \text{with} \quad \psi_a(0) = \psi_a(1) = 0, \tag{2.4}$$

where $q_a = \nabla_a^2 \psi_a := (\partial_{yy} - k_a^2)\psi_a$, c_a being the eigenvalue†. The overall pattern of the normal modes has been described by Drazin, Beaumont & Coaker (1982); here, since we consider a stable flow (e.g. assuming $\exists\alpha$ such that $Q'(U-\alpha) < 0$, $\forall y$), only two classes of modes exist:

(*a*) A discrete spectrum of regular modes with $c_a < 0$. These modes are identified as Rossby waves modified by the basic shear, and satisfy the regular differential equation

$$\frac{\mathrm{d}^2\psi_a}{\mathrm{d}y^2} + \left(\frac{Q'}{U-c_a} - k_a^2\right)\psi_a = 0. \tag{2.5}$$

For wavenumber k_a, the different modes may be distinguished by an integer n_a, so that the index a refers to the pair (k_a, n_a).

(*b*) A continuous spectrum of singular modes with $0 < c_a < 1$. These modes satisfy the singular equation

$$\frac{\mathrm{d}^2\psi_a}{\mathrm{d}y^2} + \left(\frac{Q'}{U-c_a} - k_a^2\right)\psi_a = \lambda_a\,\delta(y-y_a), \tag{2.6}$$

where λ_a is an arbitrary constant, and y_a is the critical level position defined by $U(y_a) = c_a$ (see Kamp 1991). Assuming $U'(y) > 0$, $\forall y$, the different modes can be distinguished by y_a so that a refers to (k_a, y_a).

At the critical level y_a, the streamfunction of the singular modes is continuous while its first derivative – the zonal velocity – has a logarithmic singularity. The constant λ_a is related to the jump in this derivative according to

$$\lambda_a = \left.\frac{\mathrm{d}\psi_a}{\mathrm{d}y}\right|_{y=y_a^+} - \left.\frac{\mathrm{d}\psi_a}{\mathrm{d}y}\right|_{y=y_a^-}; \tag{2.7}$$

its arbitrariness corresponds to that of the eigenfunction ψ_a, which can be multiplied by an arbitrary constant. Choosing λ_a is thus equivalent to normalizing ψ_a. Without loss of generality, we can assume that λ_a, and therefore ψ_a and q_a, are real-valued, as they will be multiplied by a complex amplitude (see (3.1) and (3.4) below).

Following Kamp (1991), and Balmforth & Morrison (1996), we interpret the modal vorticity as a distribution, and derive from (2.6) the equation

$$q_a = \lambda_a\,\delta(y-y_a) - \mathscr{P}\left(\frac{Q'\psi_a}{U-c_a}\right). \tag{2.8}$$

Here $\mathscr{P}$ signifies that a Cauchy principal value has to be taken when (2.8) is integrated. In what follows, we will perform the integration with respect to y as well as y_a; to clarify the notation we will then explicitly indicate the dependence of ψ_a, q_a, etc. on y_a. With this notation, (2.8) takes the form

$$q_a(y;y_a) = \lambda_a(y_a)\,\delta(y-y_a) - \mathscr{P}\left(\frac{Q'(y)\psi_a(y;y_a)}{U(y)-U(y_a)}\right), \tag{2.9}$$

where the subscripts a indicate the dependence on k_a. Because Cauchy principal values do not necessarily obey the same properties as usual integrals (e.g. Gakhov 1990) care needs to be exercised when (2.8)–(2.9) are integrated.

Kamp (1991) and Balmforth & Morrison (1996) proposed an alternative method to the singular differential equation (2.6) for deriving the structure of the normal modes.

† These boundary conditions are somewhat restrictive as they exclude the zonal modes with $k = 0$.

Their approach starts with the relation between vorticity and streamfunction written in the form

$$\psi_a(y) = \int_0^1 G_a(y;y')\, q_a(y')\,\mathrm{d}y', \tag{2.10}$$

where

$$G_a(y;y') = \begin{cases} -\sinh(k_a y)\sinh\left[k_a(1-y')\right]/(k_a \sinh k_a), & y < y' \\ -\sinh(k_a y')\sinh\left[k_a(1-y)\right]/(k_a \sinh k_a), & y > y' \end{cases} \tag{2.11}$$

is the Green function for ∇_a^2. Introducing (2.8) into (2.10) yields a singular integral equation for $\psi_a(y)$, namely

$$\psi_a(y) = \lambda_a G_a(y;y_a) - \mathscr{P}\int_0^1 \frac{G_a(y;y')Q'(y')\psi_a(y')}{U(y')-c_a}\,\mathrm{d}y'. \tag{2.12}$$

This equation can be regularized by normalizing the vorticity according to

$$\int_0^1 q_a(y)\,\mathrm{d}y = V, \tag{2.13}$$

where V is an arbitrarily chosen constant. It can be seen from (2.8) that V and λ_a are related through

$$\lambda_a = V + \mathscr{P}\int_0^1 \frac{Q'(y)\psi_a(y)}{U(y)-c_a}\,\mathrm{d}y, \tag{2.14}$$

so that (2.12) becomes

$$\psi_a(y) = VG_a(y;y_a) - \int_0^1 \frac{G_a(y;y') - G_a(y;y_a)}{U(y')-U(y_a)} Q'(y')\,\psi_a(y')\,\mathrm{d}y', \tag{2.15}$$

which is a regular integral equation of Fredholm type (see Tricomi 1985).

Exploiting the theory of integral equations, Balmforth & Morrison (1996) examine in great detail the general properties of the singular normal modes and their use in normal-mode expansions. They discuss specific points, such as the existence of homogeneous solutions to (2.15) (i.e. those for which $V = 0$) and the presence of unstable modes (as well as the inflection-point modes, defined by $Q'(y_a) = 0$, which constitute their limit). Here, we will ignore these aspects, assuming the sign-definiteness of Q' and hence stability. Furthermore, we will only present the results which are necessary for our purpose, namely the derivation of the equations governing the weakly nonlinear interactions, and for simplicity we will use a formulation which follows as closely as possible the treatment of the regular modes of Vanneste & Vial (1994). Proceeding this way, the interaction equations including singular modes will appear as a direct extension of those including only regular modes.

2.2. *Orthogonality relations*

The key point in the normal-mode expansion technique is the existence of orthogonality relations, on which a projection can be based. As recognized by Held (1985), these relations are related to the wave-activity conservation laws of (2.1), namely conservation of pseudoenergy and pseudomomentum. Since this is described in detail in Vanneste & Vial (1994), and Vanneste (1995) we only summarize the approach, with emphasis on the treatment of the continuous spectrum. Moreover, as orthogonality in the sense of pseudoenergy and pseudomomentum are equivalent for our purpose, only the latter will be used in what follows.

The orthogonality of modes with different wavenumber k is obvious from their sinusoidal zonal structure; we therefore temporarily focus on the meridional structure by considering two modes a and b with $k_a = k_b$. Straightforward manipulation of (2.4) yields the relation

$$(c_a - c_b) \int_0^1 q_a(y)[Q'(y)]^{-1} q_b(y)\, \mathrm{d}y = 0, \tag{2.16}$$

which is clearly related to the conservation of the pseudomomentum. Indeed, the pseudomomentum for the linearized system is given by

$$P = -\frac{1}{2} \int\!\!\int_0^1 [Q'(y)]^{-1} q^2 \, \mathrm{d}x\mathrm{d}y. \tag{2.17}$$

Orthogonality relations are derived from (2.16). For two regular modes, orthogonality is simply given by

$$\int_0^1 q_a(y)[Q'(y)]^{-1} q_b(y)\, \mathrm{d}y = -P_a\, \delta_{a,b}, \tag{2.18}$$

where $\delta_{a,b}$ is the Kronecker symbol, and $P_a = -\int_0^1 Q'^{-1} q_a^2 \mathrm{d}y$ is the pseudomomentum of mode a.

When the modes are singular, a heuristic extension of (2.18) is

$$\int_0^1 q_a(y)[Q'(y)]^{-1} q_b(y)\, \mathrm{d}y = -P_a\, \delta(y_a - y_b), \tag{2.19}$$

where $\delta(y_a - y_b)$ is the Dirac distribution. However, it is necessary to clarify the interpretation of this equation, as the integrand of the left-hand side contains singularities. Singular modes must be considered as distributions, so that they will appear in weighted integrals of explicit form

$$\int_0^1 f(y_b) q_b(y; y_b)\, \mathrm{d}y_b, \tag{2.20}$$

where $f(y_b)$ is a continuous function. Thus, for a particular mode a, the orthogonality relations will appear as the action of $\int_0^1 [Q'(y)]^{-1} q_a(y)(\cdot)\mathrm{d}y$ on this integral, i.e.

$$I = \int_0^1 [Q'(y)]^{-1} q_a(y)\, \mathrm{d}y \left[\int_0^1 f(y_b) q_b(y; y_b)\, \mathrm{d}y_b \right], \tag{2.21}$$

where the order of integration is crucial. (Where confusion may arise, we use brackets to indicate the first integral to compute.) Calculations detailed in Appendix A lead to

$$I = -P_a f(y_a), \tag{2.22}$$

where

$$P_a = -Q'(y_a) \left\{ \left[\frac{\lambda_a(y_a)}{Q'(y_a)} \right]^2 + \left[\frac{\pi \psi_a(y_a; y_a)}{U'(y_a)} \right]^2 \right\} \tag{2.23}$$

is the pseudomomentum of mode a. With this result, the orthogonality relation (2.19) can be clearly defined: in order to isolate one mode by applying (2.19) to (2.20), the order of integration in the resulting equation (2.21) *must* be permuted – regardless the singular nature of the integrand – before using the property of the Dirac distribution. Indeed, the second term on the right-hand side of (2.23) accounts for the permutation (see Appendix A). Note that when λ_a is known (e.g. if an analytical solution to the eigenvalue problem is found), the pseudomomentum of the singular modes can be evaluated without resorting to any integration.

Equation (2.19) is just one among the several orthogonality relations derived by Balmforth & Morrison (1996) (for $\beta = 0$); it corresponds to the orthogonality of the singular eigenfunctions with the eigenfunction of the adjoint equation (cf. their equation (6.17); see also Vanneste & Vial 1994). Another orthogonality relation is that related to the pseudoenergy; a third one is provided by the theory of integral equations. These relations are equivalent insofar as normalization factors such as P_a are different from 0 or ∞, which is the case with the hypothesis we made.

When a is singular and b regular, we finally remark that

$$\int_0^1 q_a(y)[Q'(y)]^{-1}q_b(y)\,\mathrm{d}y = 0.$$

This is established by noting that

$$\int_0^1 q_a(y)[Q'(y)]^{-1}q_b(y)\,\mathrm{d}y = \frac{1}{c_a - c_b}\int_0^1 [\psi_a(y)q_b(y) - \psi_b(y)q_a(y)]\,\mathrm{d}y = 0,$$

since $0 < c_a < 1$ and $c_b < 0$.

3. Interaction equations

Consider now a normal-mode expansion of the form

$$q(x,y,t) = \sum_s A_s(t)q_s(y)\exp[\mathrm{i}k_s(x - c_s t)] + \sum_r A_r(t)q_r(y)\exp[\mathrm{i}k_r(x - c_r t)], \quad (3.1)$$

where s and r denote the singular and regular modes, and where the summations are defined according to

$$\sum_s(\cdot) = \int_{-\infty}^{+\infty}\int_0^1 (\cdot)\,\mathrm{d}k_s\mathrm{d}y_s, \quad \text{and} \quad \sum_r(\cdot) = \int_{-\infty}^{+\infty}\sum_{n_r}(\cdot)\,\mathrm{d}k_r, \quad (3.2)$$

in the case of an infinite channel. If the channel is periodic, $\sum_k$ should replace $\int_{-\infty}^{+\infty}\mathrm{d}k$. The reality of q is ensured by considering together the physically equivalent modes a (k_a, c_a) and $-a$ $(-k_a, c_a)$ with $A_{-a}(t) = [A_a(t)]^\star$. First note that the expansion (3.1) is an exact solution of the linearization of (2.1) with $A_a(t) = A_a(0)$ for each mode a (the amplitudes can be derived from the initial conditions using the orthogonality relations). This can be used to show in a straightforward fashion that the pseudomomentum P is an exact invariant for the linear system; indeed, introducing (3.1) into (2.17), and using (2.18)–(2.19) we obtain

$$P = \frac{1}{2}\left(\sum_s P_s\,|A_s|^2 + \sum_r P_r\,|A_r|^2\right). \quad (3.3)$$

For the nonlinear system, P is the lowest-order (quadratic) part of the pseudomomentum discovered by Killworth & McIntyre (1985), and it can be shown that $\mathrm{d}P/\mathrm{d}t = O(A^3)$.

For applications, using the expansion of the streamfunction rather than the expansion the vorticity is often preferable, since the first contains no Cauchy principal value. However, owing to the presence of a singular integral in the expansion (3.1) (see (2.8)), it may not be obvious whether the streamfunction, implicitly defined by

$\nabla^2\psi = q$, can be expanded in a similar fashion as (3.1). That this is in fact the case can be checked by noting that for the packet of singular modes corresponding to each wavenumber k_a,

$$\begin{aligned}
\psi(y) &= \int_0^1 G_a(y;y')q(y')\,\mathrm{d}y' \\
&= \int_0^1 G_a(y;y')\,\mathrm{d}y' \left[\int_0^1 A_a(t;y_a)q_a(y';y_a)\exp\left[\mathrm{i}k_a\,(x-U(y_a)t)\right]\,\mathrm{d}y_a\right] \\
&= \int_0^1 A_a(t;y_a)\exp\left[\mathrm{i}k_a\,(x-U(y_a)t)\right]\,\mathrm{d}y_a \left[\int_0^1 G_a(y;y')q_a(y';y_a)\,\mathrm{d}y'\right] \\
&= \int_0^1 A_a(t;y_a)\psi_a(y;y_a)\exp\left[\mathrm{i}k_a\,(x-U(y_a)t)\right]\,\mathrm{d}y_a.
\end{aligned}$$

Here, the change of the order of integration is allowed as only one integral is singular (e.g. Gakhov 1990). Therefore, we can write

$$\psi(x,y,t) = \sum_s A_s(t)\psi_s(y)\exp[\mathrm{i}k_s(x-c_st)] + \sum_r A_r(t)\psi_r(y)\exp[\mathrm{i}k_r(x-c_rt)]. \tag{3.4}$$

Once the streamfunction is calculated, the vorticity is derived from $\nabla^2\psi = q$.

Introducing the expansions (3.1) and (3.4) in the governing equation (2.1), and taking into account the orthogonality relations (2.18) and (2.19), we can transform the partial differential equation into a infinite set of ordinary differential equations. This set is discrete in the regular modes ($a = r$) and continuous in the singular modes ($a = s$). These equations can be *formally* written as

$$\frac{\mathrm{d}A_a}{\mathrm{d}t} = \frac{\mathrm{i}}{2}\sum_b\sum_c I_a^{bc}A_b^\star A_c^\star\,\exp(\mathrm{i}\Omega_{abc}t), \tag{3.5}$$

where $\Omega_{abc} = \omega_a + \omega_b + \omega_c$ is a detuning parameter, and

$$I_a^{bc} = \mathrm{i}P_a^{-1}\int_0^1 q_a(y)[Q'(y)]^{-1}\left[\partial(\psi_b,q_c) + \partial(\psi_c,q_b)\right]\,\mathrm{d}y \tag{3.6}$$

is the interaction coefficient. Here, $\partial(\psi_b,q_c)$ designates a Jacobian in which (2.3) has been introduced, i.e. $\partial(\psi_b,q_c) = \mathrm{i}k_b\psi_bq_{c,y} - \mathrm{i}k_c\psi_{b,y}q_c$, so that I_a^{bc} is real. The right-hand side of (3.5) includes all modes satisfying the interaction condition

$$k_a + k_b + k_c = 0, \tag{3.7}$$

and thus includes both regular and singular modes, the corresponding summations being those defined in (3.2).

Equations (3.5)–(3.6) are similar to those derived by Ripa (1981) for waves in a resting medium, and by Vanneste & Vial (1994) for regular waves in a shear flow. However, the interaction coefficient (3.6) is purely formal when b and/or c are singular modes. Therefore, an interpretation of (3.5) is necessary, i.e. we must state the order in which the integrations contained in (3.5) are to be carried out. This order is fixed by the procedure leading to (3.5): packets of modes b and c are first introduced in the nonlinear term of (2.1) before the equation is projected on mode a in order to isolate the evolution of A_a. Integrations with respect to y_b and y_c are thus performed before the integration with respect to y. Therefore, when evaluating (3.5)–(3.6), the order of integration *must* be permuted for the evolution equation of A_a to be correct; this rule can be seen as part of the definition of I_a^{bc}.

For instance, consider a triad a, b, c, where a and c are regular modes, and b corresponds to a packet of singular modes such that (3.7) is satisfied. In this particular case, the contribution of b and c to $\mathrm{d}A_a/\mathrm{d}t$ takes the explicit form

$$-\frac{\mathrm{i}}{2}P_a^{-1}A_c^{\star}\exp\left[\mathrm{i}(\omega_a+\omega_c)t\right]\int_0^1 q_a Q'^{-1}\left(k_b\tilde{\psi}^{\star}q_{c,y}-k_c\tilde{\psi}_y^{\star}q_c+k_c\psi_c\tilde{q}_y^{\star}-k_b\psi_{c,y}\tilde{q}^{\star}\right)\,\mathrm{d}y, \tag{3.8}$$

where

$$\tilde{\psi} := \int_0^1 A_b(t;y_b)\psi_b(y;y_b)\exp\left[-\mathrm{i}k_b U(y_b)t\right]\,\mathrm{d}y_b,$$

and

$$\tilde{q} := \int_0^1 A_b(t;y_b)q_b(y;y_b)\exp\left[-\mathrm{i}k_b U(y_b)t\right]\,\mathrm{d}y_b = \nabla_b^2\tilde{\psi}$$

are the streamfunction and vorticity of the entire packet of singular modes with wavenumber k_b. This expression shows what is meant by (3.5)–(3.6); the latter equations may thus be regarded as a shorthand notation for a lengthy expression that is obtained through a conventional permutation of operators. Once the complete expression is written down, further changes of the order of integration can be carried out, but the permissibility of such operations must be checked (the Poincaré–Bertrand transposition formula must be used when two singular integrals are permuted (e.g. Gakhov 1990)).

When a is a singular mode, its time derivative will always contain a Cauchy principal value, owing to the projection $\dot{\int}_0^1 q_a Q'^{-1}(\cdot)\mathrm{d}y$. In the expansions (3.1)–(3.4), the amplitude $A_a(t)$ is never considered as isolated, but rather appears inside integrals over y_a. As will become clear in the next section, this can be exploited to replace the principal value by an ordinary integral.

In principle, the time evolution of the mode amplitudes $A_a(t)$ could be obtained by solving the coupled system of equations (3.5), and then the vorticity and streamfunction could be deduced from (3.1) and (3.4). However, the use of the normal-mode expansion including singular modes presents formidable technical difficulties if a large number of modes are retained, and it is undoubtedly not suited for efficient numerical simulations. Standard spectral methods, which are based on complete sets of orthogonal modes do not yield exact solutions of the linearized equations; however, by contrast with the normal modes, these sets are countable, and this certainly constitutes an advantage for numerical applications. Nevertheless, insight into the weakly nonlinear behaviour of a model may be gained by studying severe truncations of the equations (3.5). When truncating the system, it must be kept in mind that a complete packet of singular modes must be taken into account as soon as one singular mode with the corresponding wavenumber is selected. The members of a wave triad are therefore either regular modes or complete packets of singular modes.

4. Interaction of two regular modes with a packet of singular modes

We now turn to the detailed analysis of the formation of a packet of singular modes by two interacting regular modes. Let a designate all the singular modes of zonal wavenumber k_a, and let b and c designate two regular modes, satisfying the interaction condition (3.7). For moderate mode amplitudes, the interaction is significant only if b and c constitute a resonant triad with a particular singular mode of the packet a,

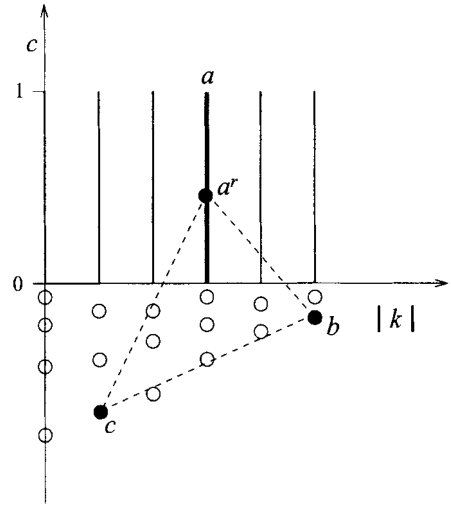

FIGURE 1. A schematic view of the interaction between two regular modes and a packet of singular modes for a periodic channel. The dispersion relation is represented in the (k, c)-plane: the circles with $c < 0$ denote the regular modes, and the vertical lines with $0 < c < 1$ denote the continuum of singular modes corresponding to each zonal wavenumber. The interaction involves the two regular modes b and c (solid circles), and the entire packet of singular modes a (thick vertical line). A particular singular mode a^r constitutes a resonant triad (represented by the dashed triangle) with b and c.

denoted by the superscript r, i.e.

$$\omega_a^r + \omega_b + \omega_c = 0 \tag{4.1}$$

(see figure 1). Together with the fact that the resonant mode (with wavenumber k_a and frequency ω_a^r) belongs to the continuous spectrum of singular modes, this equation dictates a condition on b and c, namely

$$0 < c_a^r = \frac{\omega_b + \omega_c}{k_b + k_c} < 1. \tag{4.2}$$

Note that, because of the continuous character of the singular mode spectrum, resonant triads including singular modes are much more numerous than resonant triads including regular modes only. For simplicity, we assume $A_a(0) = 0$, and $|A_b|, |A_c| \gg |A_a|$ for all $0 < y_a < 1$. It follows from the latter assumption that the amplitudes A_b and A_c remain approximately constant (i.e. the feedback of the singular modes on the regular modes can be neglected); therefore we can focus on the evolution of the packet of singular modes a. The evolution of A_b and A_c could, however, be obtained without conceptual difficulties, as described in the previous section. Besides its illustrative interest for the theoretical concepts of §3, the very simple problem examined here can be viewed as a paradigm for the interactions between singular and regular modes: the entire packet of singular modes must be retained in the truncated system. While it could be thought that the interaction of two regular modes with singular modes can be analysed solely in terms of resonant interactions, as is done with regular modes, we shall show that the non-resonant interactions play a fundamental rôle: they act in such a way that the streamfunction and the vorticity of the packet remain smooth functions of y for all times.

The evolution equation for the amplitude A_a is given by

$$\frac{\mathrm{d}A_a}{\mathrm{d}t} = \mathrm{i}I_a^{bc}A_b^\star A_c^\star \exp(\mathrm{i}\Omega_{abc}t),$$

and is readily integrated to yield

$$A_a(t) = I_a^{bc}A_b^\star A_c^\star \frac{\exp\left[\mathrm{i}\left(\omega_a - \omega_a^r\right)t\right] - 1}{\omega_a - \omega_a^r}, \tag{4.3}$$

where we have taken into account (4.1) and the initial condition $A_a(0) = 0$. From (3.4), one sees that the streamfunction can be written as (twice the real part of)

$$\psi = A_b^\star A_c^\star \exp\left[ik_a\left(x - c_a^r t\right)\right]\, \tilde{\psi}(y,t), \tag{4.4}$$

where the meridional and temporal structure depends on $\tilde{\psi}$, given by

$$\tilde{\psi}(y,t) = \int_0^1 I_a^{bc}\psi_a(y;y_a)g_a(t;y_a)\,\mathrm{d}y_a, \tag{4.5}$$

with

$$g_a(t;y_a) := \frac{1 - \exp\left[-\mathrm{i}\left(\omega_a - \omega_a^r\right)t\right]}{\omega_a - \omega_a^r}.$$

Using the definition (3.6), we find

$$\tilde{\psi}(y,t) = \int_0^1 [P_a(y_a)]^{-1}\psi_a(y;y_a)g_a(t;y_a)\mathrm{d}y_a \left[\int_0^1 q_a(y';y_a)f_{bc}(y')\mathrm{d}y'\right], \tag{4.6}$$

where

$$f_{bc}(y') := \mathrm{i}[Q'(y')]^{-1}\left[\partial(\psi_b, q_c) + \partial(\psi_c, q_b)\right]$$

depends solely on the regular modes. As (4.6) contains a single singularity (namely that of $q_a(y';y_a)$), the order of integration can be changed. Further, the Cauchy principal value can be avoided by integrating by parts with respect to y' and noting that $f_{bc}(0) = f_{bc}(1) = 0$ to obtain

$$\tilde{\psi}(y,t) = \int_0^1 \nabla_a'^2 f_{bc}(y')\,\mathrm{d}y' \left[\int_0^1 [P_a(y_a)]^{-1}\psi_a(y;y_a)\psi_a(y';y_a)g_a(t;y_a)\mathrm{d}y_a\right], \tag{4.7}$$

where $\nabla_a'^2 := \partial_{y'y'} - k_a^2$. The latter integral only involves regular terms and can thus be evaluated accurately by standard methods, as is done in the next section. Together with (4.4), it gives a convenient form for the streamfunction, from which the vorticity is derived by applying ∇_a^2. The pseudomomentum of the packet of singular modes a is a useful physical quantity: in the linear approximation, it is quadratic and conserved, and it is dissociated from the pseudomomentum of regular modes with the same wavenumber. Collecting (3.3), (3.6), and (4.3), and applying integration by parts yields this pseudomomentum in the form

$$P = |A_bA_c|^2 \int_0^1 [P_a(y_a)]^{-1}\,|g_a(t;y_a)|^2\,\mathrm{d}y_a \left[\int_0^1 \psi_a(y';y_a)\nabla_a'^2 f_{bc}(y')\,\mathrm{d}y'\right]^2. \tag{4.8}$$

It is fruitful to examine the asymptotic behaviour of the packet streamfunction and vorticity for $t \ll 1$ and $t \gg 1$. This behaviour is obviously governed by the form of $g_a(t;y_a)$. For $t \ll 1$, $g_a(t;y_a) \approx \mathrm{i}t$ does not depend on the mode y_a; hence all modes grow simultaneously. The streamfunction and the vorticity of the packet of singular modes then increase linearly with time, while the pseudomomentum evolves like t^2. This evolution is the same as that experienced by a regular mode forced by resonant interaction, as long as its influence on the other two members of the triad remains negligible. As time increases, $g_a(t; y_a)$ peaks around the resonant mode (with phase velocity c_a^r and critical level at y_a^r), and the evolution of the packet slows down. For sufficiently small amplitudes of the regular waves A_b and A_c, there is a range of t in which the condition $t \gg 1$ is consistent with the assumption of constant of amplitudes†. Calculations detailed in Appendix B provide the asymptotic form of the streamfunction,

$$\tilde{\psi} \approx \frac{1}{k_a P_a(y_a^r)} \psi_a(y; y_a^r) \left(\ln \frac{1 - c_a^r}{c_a^r} + \mathrm{i}\pi \right) \int_0^1 \psi_a(y'; y_a^r) \nabla'^2_a f_{bc}(y')\, \mathrm{d}y'$$
$$+ \int_0^1 \nabla'^2_a f_{bc}(y')\, \mathrm{d}y' \int_0^1 \frac{1}{\omega_a - \omega_a^r} \left[\frac{\psi_a(y; y_a)\psi_a(y', y_a)}{P_a(y_a)} - \frac{\psi_a(y; y_a^r)\psi_a(y', y_a^r)}{P_a(y_a^r)} \right] \mathrm{d}y_a. \tag{4.9}$$

Together with (4.4), this equation indicates that the streamfunction (and equivalently the velocity and the vorticity) tends to a steady wave propagating with the phase velocity of the resonant mode, c_a^r. Perhaps surprisingly, the corresponding meridional structure is not simply given by the meridional structure of the resonant mode (first term on the right-hand side of (4.9)), but it also contains a smooth contribution (second term on the right-hand side of (4.9)), which involves the entire packet of singular modes. This latter contribution is in fact only piecewise smooth, the vorticity undergoing a jump at y_a^r. Near y_a^r, the asymptotic zonal velocity and vorticity are dominated by the structure of the resonant mode, and thus have singularities of the form $\ln|y - y_a^r|$ and $1/(y - y_a^r)$, respectively. As will clearly appear in the particular case presented in the next section, the zonal velocity is smooth for finite time, but it develops a peak at y_a^r which sharpens as time increases. (There is an analogous behaviour for the vorticity.) Of course, the hypothesis of weak nonlinearity we have made breaks down when the amplitudes become too important; to study the long-term evolution of the flow, the action of the singular packet on the regular modes must be taken into account. If the initial amplitudes of the regular modes are very strong, the system (3.5) should be considered with a less severe truncation than the one adopted here, and possibly viscosity needs to be included.

The pseudomomentum for $t \gg 1$ is given at leading order by (see Appendix B)

$$P \approx \frac{2\pi t}{k_a P_a(y_a^r)} |A_b A_c|^2 \left[\int_0^1 \psi_a(y'; y_a^r) \nabla'^2_a f_{bc}(y')\, \mathrm{d}y' \right]^2. \tag{4.10}$$

The magnitude of the pseudomomentum of the packet thus increases linearly with time, i.e. slower than the pseudomomentum of a regular mode in a similar situation. The unbounded growth of P is consistent with the fact that the asymptotic state given by (4.9) has a singular vorticity structure, and hence an infinite pseudomomentum. Note that by contrast with (4.9), (4.10) depends only on the structure of the resonant mode.

† Specifically, the asymptotic expansion is valid for $t \gg 1/\min(|\omega_a^r|, |k_a - \omega_a^r|)$ (i.e. for $t \gg 1/[|k_a| \min(c_a^r - U_m, U_M - c_a^r)]$, see Appendix B); on the other hand, A_b and A_c are approximately constant for $t \ll 1/\max(|A_b|^2, |A_c|^2)$. Therefore, for sufficiently small amplitudes A_b and A_c, both conditions can be satisfied.

5. Application to a linear shear flow

To illustrate the general results of the previous section, we investigate the particular case of a linear shear $U(y) = y$ in a channel. In this case, $Q' = \beta$ and the critical level location is simply $y = c$. As most previous work has concerned unbounded or semi-bounded domains (see Tung 1983), or does not focus on the eigenvalue problem, we now present analytical solutions for both the regular and the singular normal modes. Although these analytical expressions are not required to apply the formulae of §§2–4, they permit a direct calculation of the streamfunction and vorticity for arbitrarily chosen values of the critical level location, and hence an accurate evaluation of the required integrals. Furthermore, we establish an analytical expression for the pseudomomentum of the singular modes, which enable us to avoid singular integral evaluations.

5.1. *Normal modes*

Consider first the regular modes, i.e. those with $c < 0$. They satisfy the eigenvalue problem (2.5), which takes the form

$$\frac{d^2\psi}{dy^2} + \left(\frac{\beta}{y-c} - k^2\right)\psi = 0, \quad \text{with } \psi(0) = \psi(1) = 0, \tag{5.1}$$

where we have temporarily simplified the notation by suppressing the mode index. Without loss of generality, we may assume $k > 0$ in the calculation of the eigensolution. Introducing the new variable

$$z_c := 2k(y-c)$$

depending on both y and c, the solution to (5.1) may be expressed in terms of confluent hypergeometric functions, or equivalently in terms of Whittaker functions (e.g. Erdélyi *et al.* 1953; Slater 1960) according to

$$\psi = A W_{\kappa,1/2}(z_c) + B M_{\kappa,1/2}(z_c), \tag{5.2}$$

where $\kappa := \beta/(2k)$. Note that the solutions involved in (5.2) are linearly dependent when κ is an integer (Erdélyi *et al.* 1953); therefore, we assume that κ is non-integer so as to avoid difficulties associated with the definition of the branch cut in the fundamental system of solutions given in Erdélyi *et al.* (1953). Analytic continuation of the final results could be invoked to deal with this particular case. Denoting by $z_{c,0} := z_c(0)$ and $z_{c,1} := z_c(1)$ the boundary values of z_c, (5.2) leads to a non-trivial solution to the eigenvalue problem (5.1) when

$$D(c,k) := \begin{vmatrix} W_{\kappa,1/2}(z_{c,0}) & M_{\kappa,1/2}(z_{c,0}) \\ W_{\kappa,1/2}(z_{c,1}) & M_{\kappa,1/2}(z_{c,1}) \end{vmatrix} = 0. \tag{5.3}$$

This is an implicit form of the dispersion relation, which gives for each wavenumber k a finite number of phase velocities $c < 0$. When (5.3) is satisfied, the solutions of (5.1) can be written

$$\psi = A\left[M_{\kappa,1/2}(z_{c,0})W_{\kappa,1/2}(z_c) - W_{\kappa,1/2}(z_{c,0})M_{\kappa,1/2}(z_c)\right]. \tag{5.4}$$

Consider now the singular modes, i.e. those with $0 < c < 1$, with critical level at $z_c = 0$. They satisfy the singular eigenvalue problem

$$\frac{d^2\psi}{dy^2} + \left(\frac{\beta}{y-c} - k^2\right)\psi = \lambda\,\delta(y-c), \quad \text{with } \psi(0) = \psi(1) = 0, \tag{5.5}$$

whose solution can be written in a piecewise form as

$$\psi = \begin{cases} A^- M_{\kappa,1/2}(z_c) + B^- W_{-\kappa,1/2}(-z_c), & y < c \\ A^+ M_{\kappa,1/2}(z_c) + B^+ W_{\kappa,1/2}(z_c), & y > c \end{cases}.$$

The continuity of the pressure across the critical level can be written in terms of the variable z_c as

$$\psi(0^-) = \psi(0^+). \tag{5.6}$$

It implies that

$$\Gamma(1-\kappa)B^- = \Gamma(1+\kappa)B^+.$$

The constants A^- and A^+ are related to B^- and B^+, respectively, when the boundary conditions are taken into account. Finally, the streamfunction of the singular modes is given by

$$\psi = \begin{cases} A\,\Gamma(1+\kappa)M_{\kappa,1/2}(z_{c,1})\left[W_{-\kappa,1/2}(-z_{c,0})M_{\kappa,1/2}(z_c) \right. \\ \quad \left. -M_{\kappa,1/2}(z_{c,0})W_{-\kappa,1/2}(-z_c)\right], & y < c \\ A\,\Gamma(1-\kappa)M_{\kappa,1/2}(z_{c,0})\left[W_{\kappa,1/2}(z_{c,1})M_{\kappa,1/2}(z_c) \right. \\ \quad \left. -M_{\kappa,1/2}(z_{c,1})W_{\kappa,1/2}(z_c)\right], & y > c \end{cases}. \tag{5.7}$$

Near the critical level, the corresponding zonal velocity and vorticity are singular like $\ln|z_c|$ and $1/z_c$, respectively.

It is essential to determine the value of λ corresponding to the solution (5.7). Calculations detailed in Appendix C give

$$\lambda = -A\beta\,\eta, \tag{5.8}$$

where η is complicated expression given by (C 1). The asymptotic expansion of the Whittaker functions also provides the value of the streamfunction at the critical level:

$$\psi(c) = -AM_{\kappa,1/2}(z_{c,0})M_{\kappa,1/2}(z_{c,1}).$$

Note that because of the β-effect this expression may vanish for certain c. Introducing these results in (2.23), we get an analytical expression for the pseudomomentum of the singular modes in the form

$$P = -A^2\beta\left[\eta^2 + \pi^2 M^2_{\kappa,1/2}(z_{c,0})M^2_{\kappa,1/2}(z_{c,1})\right]. \tag{5.9}$$

We have verified for several different cases that the analytical results (5.7)–(5.9) are consistent with the numerical solutions derived from the integral-equation approach of Kamp (1991) and Balmforth & Morrison (1996). In this approach, λ is given by a singular integral, according to (2.14).

5.2. *Nonlinear interaction*

We present here explicit results on the action of two regular modes b and c on a packet of singular modes a in a linear shear. As a typical case, we choose $\beta = 40$ (weak shear); the selected wavenumbers are

$$k_a = 2.1, \quad k_b = -2.7, \quad k_c = 0.6,$$

which satisfy the interaction condition (3.7). The streamfunctions of the singular modes with $0 < c_a < 1$ are displayed in figure 2 for four distinct values of c_a. Regular modes are defined not only by their zonal wavenumber, but also by the number of zeros of the streamfunction in the range $y \in]0,1[$. We consider here the mode b with

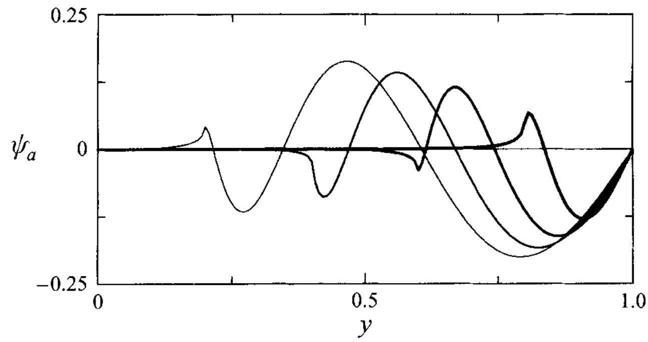

FIGURE 2. Streamfunction of the singular modes in the linear shear with $\beta = 40$, for $k_a = 2.1$, and $c_a = 0.2, 0.4, 0.6, 0.8$ (with increasing line width). The normalization corresponds to $P_a = -1$.

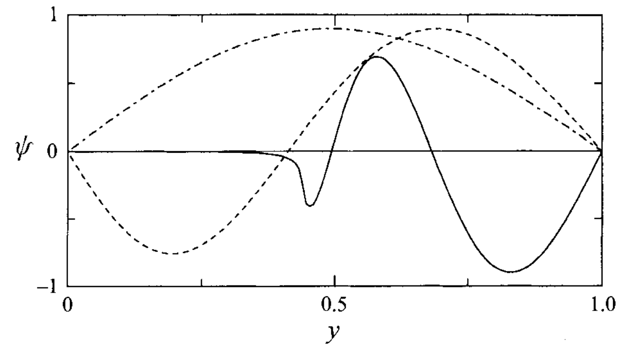

FIGURE 3. Streamfunction of the resonant singular mode ($k_a = 2.1, \omega_a^r = 0.918$; solid curve), and of the regular modes b ($k_b = -2.7, \omega_b = 1.135$; dashed curve) and c ($k_c = 0.6, \omega_c = -2.053$; dash-dotted curve), with arbitrary normalization.

one zero, and the mode c without a zero. Numerically solving for the corresponding roots of (5.3) yields the frequencies

$$\omega_b = 1.135, \quad \omega_c = -2.053,$$

so that (4.2) is satisfied with $c_a^r = 0.437$, and thus $\omega_a^r = 0.918$. Figure 3 displays the streamfunctions of the regular modes b and c, and of the resonant singular mode a^r.

To calculate expressions (4.7)–(4.10), the first step is the evaluation of the pseudomomentum density P_a according to (5.9). An accurate calculation of this density turns out to be crucial for the final results: in some sense (see Ripa 1990), P_a corresponds to the weight of each singular mode in the packet. We have chosen to directly normalize the streamfunction so that $P_a = -1$ for all the singular modes; this is done by a proper choice of the arbitrary amplitudes A_a, according to (5.9). (Note that the pseudomomentum of the singular modes is negative as $\beta > 0$; their pseudoenergy $E_a = c_a P_a$ is therefore negative, by contrast with that of the regular modes.) While the final results are independent of the (arbitrary) normalization chosen for the singular modes, they do depend on the physical amplitude of the regular modes, which can be measured by their pseudomomenta $P_b|A_b|^2$ and $P_c|A_c|^2$. As this dependence consists of a simple scaling, we take $P_b = P_c = -1$ and $A_b = A_c = 1$.

The streamfunction of the packet of singular modes, given by (4.7), can then be computed for any time t. The numerical integration is carried out using a uniform

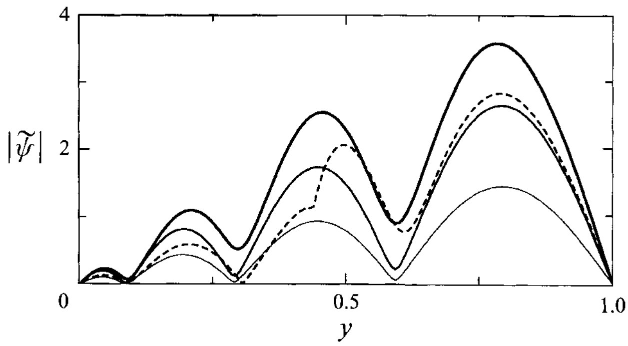

FIGURE 4. Amplitude of the streamfunction of the packet of singular modes for $t = 1, 2, 5$ (with increasing line width), and $t \gg 1$ (dashed curve).

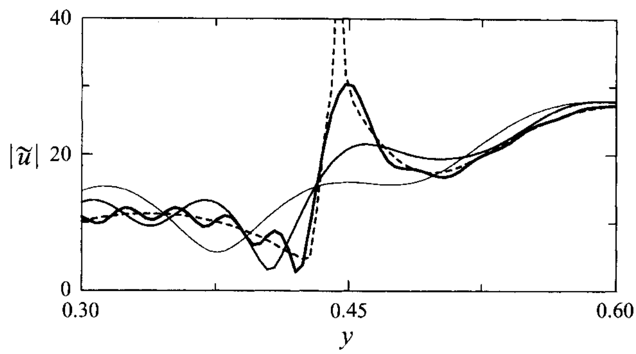

FIGURE 5. Amplitude of the zonal velocity of the packet of singular modes for $t = 20, 40, 100$ (with increasing line width), and $t \gg 1$ (dashed curve).

grid in the (y, y_a)-plane (typically 150×150). We have confirmed that the results are insensitive to the grid size. Figure 4 presents this streamfunction for $t = 1, 2, 5$, as well as the asymptotic steady state (4.9) obtained for $t \gg 1$. For finite times, the streamfunction is smooth, while for $t \to \infty$, it possesses a vertical tangent at the critical level of the resonant mode $y = c_a^r$, as explained in §4. In fact, after $t \approx 20$ the streamfunction is close to its asymptotic structure almost everywhere, except in a narrow vicinity of $y = c_a^r$, where it steepens continually and only slowly tends to the asymptotic structure. Correspondingly, the zonal velocity and vorticity peak at $y = c_a^r$, with an increasing amplitude of the peak. This can be seen from figure 5, which shows the amplitude of the zonal velocity $\tilde{u} = \mathrm{d}\tilde{\psi}/\mathrm{d}y$ at $t = 20, 40, 100$, and for $t \to \infty$ in the vicinity of the critical level of the resonant mode.

The pseudomomentum, which is simply proportional to the enstrophy, can be calculated from (4.8), and its evolution is displayed in figure 6. The asymptotic results for $t \ll 1$ (which is obtained by introducing the approximation $g_a(t; y_a) \approx \mathrm{i}t$ in (4.8)) and for $t \gg 1$ are also shown, and can be seen to provide useful estimates for the pseudomomentum. We have also calculated the pseudomomentum from its basic form (2.17); this constitutes a good check for (4.8).

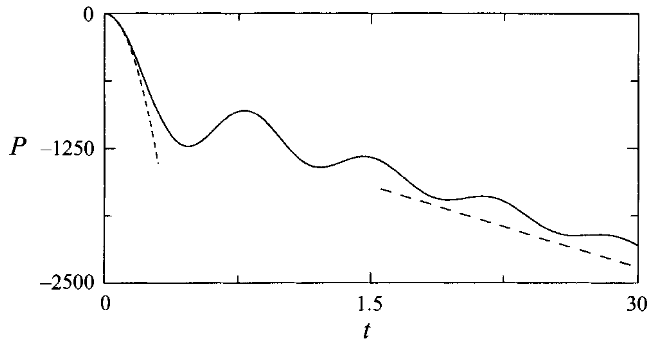

FIGURE 6. Evolution of the pseudomomentum of the packet of singular modes (solid curve). The quadratic approximation for $t \ll 1$ (short-dashed curve) and the asymptotic slope for $t \gg 1$ (long-dashed curve) are also displayed.

6. Discussion

Interactions in triads of dispersive waves constitute an important process in the weakly nonlinear dynamics of fluid flows, and in spite of the considerable simplifications they imply, their significance is well established, in particular for geophysical flows. As the presence of a basic shear flow modifies the properties of the waves, a natural question concerns the way it affects the properties of the interactions. Part of the answer is provided by work on layered models (e.g. Cairns 1979; Craik & Adam 1979; Vanneste 1995), and recent work on continuous models (Becker & Grimshaw 1993; Vanneste & Vial 1994). The rôle of the continuous spectrum of singular modes present in continuously sheared flows, which seems to have received little attention in the context of triad interactions, is addressed in this paper. Specifically, it is shown that on the basis of a simple model of Rossby waves, the normal-mode expansion, conventionally used for waves propagating in a resting basic state, is also suitable for dealing with the continuous spectrum of singular modes. The essential point, justified both mathematically and physically, is that complete packets of singular modes of a given wavenumber need to be considered. The advantages of the normal-mode expansion (Ripa 1981) are thus preserved in the presence of a continuous shear and its associated spectrum of singular modes, even though some technical difficulties will arise. The properties of the singular normal modes in the flow considered have been studied in depth by Kamp (1991) and Balmforth & Morrison (1996); our work can be viewed as an extension of their results to the nonlinear domain.

The interaction of two regular modes with a packet of singular modes is treated in detail to illustrate the usefulness of the approach. With the assumption of constant amplitude for the regular modes, this problem may be regarded as a forced problem for the packet of singular modes; but it is very different from that usually studied (see Maslowe 1986), as the forcing is not limited to a boundary. The results clarify the crucial rôle of the non-resonant interactions, as can be seen in particular from the asymptotic result (4.9). To deal with the full initial-value problem set by the interaction of a triad consisting of two regular modes and a packet of singular modes, the coupled action of a regular mode and the singular packet on the second regular mode should also be taken into account. This is certainly feasible using our approach, but would require a substantial amount of calculation. In the nonlinearly stable flow we have analysed, the feedback of the singular packet on the regular modes can be expected to reduce the amplitude of the latter, and therefore to limit the development of the critical layer obtained when the assumption of constant amplitude is made for the regular modes.

Various problems in which the rôle of the continuous spectrum of singular modes deserves investigation can be suggested. A first one concerns the stability of regular waves to disturbances consisting partly of singular modes. A second one, mentioned in the Introduction as a motivation for the present work, concerns the explosive resonant interaction and the related nonlinear instability of shear flows. In addition to Becker & Grimshaw's result (1993) which reveals the necessity of singular modes for explosive interaction, it may be pointed out that the relevance of singular modes to certain instabilities of stratified flows is strongly suggested by the fact that disturbances with small vertical scale preclude nonlinear stability of such flows (Abarbanel *et al.* 1986; see Ripa 1990; Shepherd 1992). For triads of regular modes, these two problems are mainly analysed in terms of resonant interaction. In resonant triads, the quadratic parts of the pseudoenergy and of the pseudomomentum (explicitly, $P_a|A_a|^2+P_b|A_b|^2+P_c|A_c|^2$) are conserved. Consequently, relations among the interactions coefficients, the wavenumbers and the mode pseudomomenta (or equivalently, the frequencies and the mode pseudoenergies) are found, which constitute the basis of the criteria for wave instability and explosive interaction (see Ripa 1981; Vanneste & Vial 1994). The extension of these criteria to singular modes is not straightforward: non-resonant interactions must then be taken into account, so that the quadratic parts of the pseudoenergy and of the pseudomomentum are not necessarily conserved, and the interaction properties become more complex. As an example of this complexity, it can be mentioned that singular modes with positive and negative pseudomomentum exist for any wavenumber if $Q(y)$ is non-monotonic (see (2.23)). Any packet of singular modes will thus systematically involve modes with pseudomomentum of both signs. Clearly, much progress still needs to be made for completely understanding the rôle of the continuous spectrum in nonlinear interactions, and we hope the approach presented in this paper will prove useful in this respect.

The author wishes to thank G. Brunet for fruitful discussions. Comments of K. Ngan, and P. Ripa on an early version of the manuscript are much appreciated. Part of the work reported in this paper was carried out at the Laboratoire de Météorologie Dynamique du CNRS, where the author enjoyed stimulating conversations with F. Lott, H. Teitelbaum, and F. Vial. The support from a NATO fellowship is gratefully acknowledged.

Appendix A. Orthogonality of singular modes

Introducing (2.9) in (2.21) yields

$$
\begin{aligned}
I = {} & [Q'(y_a)]^{-1}[\lambda_a(y)]^2 f(y_a) \\
& -\lambda_a(y_a)\,\mathscr{P}\int_0^1 \frac{f(y_b)\psi_b(y_a;y_b)}{U(y_a)-U(y_b)}\,\mathrm{d}y_b - \mathscr{P}\int_0^1 \lambda_b(y)\frac{f(y)\psi_a(y;y_a)}{U(y)-U(y_a)}\,\mathrm{d}y \\
& +\mathscr{P}\int_0^1 \frac{Q'(y)\psi_a(y;y_a)}{U(y)-U(y_a)}\,\mathrm{d}y\,\left[\mathscr{P}\int_0^1 \frac{f(y_b)\psi_b(y;y_b)}{U(y)-U(y_b)}\,\mathrm{d}y_b\right].
\end{aligned} \tag{A 1}
$$

The first two integrals can be combined by changing the integration variable in the second from y to y_b to give

$$-\int_0^1 \frac{f(y_b)}{U(y_a)-U(y_b)}\,\mathrm{d}y_b \int_0^1 [\lambda_a(y_a)\psi_b(y;y_b)\delta(y-y_a)-\lambda_b(y_b)\psi_a(y;y_a)\delta(y-y_b)]\,\mathrm{d}y. \tag{A 2}$$

Following Balmforth & Morrison (1996), we can use the Poincaré–Bertrand transposition formula (e.g. Gakhov 1990) to change the order of integration in the last integral. It now becomes

$$\left[\frac{\pi\psi_a(y_a;y_a)}{U'(y_a)}\right]^2 Q'(y_a)f(y_a)+\int_0^1 f(y_b)\,\mathrm{d}y_b\left[\mathscr{P}\int_0^1 \frac{Q'(y)\psi_a(y;y_a)\psi_b(y;y_b)}{[U(y)-U(y_a)][U(y)-U(y_b)]}\,\mathrm{d}y\right]. \tag{A 3}$$

Note that the integration with respect to y_b is now defined in the usual sense, so that no $\mathscr{P}$ symbol is needed. Gathering (A 2)–(A 3), we see that the integrals cancel, as their sum is

$$-\int_0^1 \frac{f(y_b)}{U(y_a)-U(y_b)}\,\mathrm{d}y_b \int_0^1 [q_a(y;y_a)\psi_b(y;y_b)-q_b(y;y_b)\psi_a(y;y_a)]\,\mathrm{d}y, \tag{A 4}$$

which vanishes after integration by parts. Introducing this result in (A 1), we finally obtain

$$I = Q'(y_a)\left\{\left[\frac{\lambda_a(y_a)}{Q'(y_a)}\right]^2+\left[\frac{\pi\psi_a(y_a;y_a)}{U'(y_a)}\right]^2\right\}f(y_a). \tag{A 5}$$

Appendix B. Asymptotic results for $t \gg 1$

Consider the limit of the second integral involved in (4.7) for $t\to\infty$, i.e.

$$\lim_{t\to\infty}\int_0^1 [P_a(y_a)]^{-1}\psi_a(y;y_a)\psi_a(y';y_a)g_a(t;y_a)\,\mathrm{d}y_a.$$

Introducing $v := \omega_a-\omega_a^r$, and $h := [k_aP_a(y_a)]^{-1}\psi_a(y;y_a)\psi_a(y';y_a)$ which can be viewed as a function of v, this limit takes the form

$$\lim_{t\to\infty}\int_{v_m}^{v_M} h(v)\left[\frac{1-\cos(vt)}{v}+\mathrm{i}\frac{\sin(vt)}{v}\right]\mathrm{d}v, \tag{B 1}$$

where the boundaries are $v_m=-\omega_a^r$ and $v_M=k_a-\omega_a^r$. Without loss of generality we may assume $k_a>0$, so that $v_m<0$ and $v_M>0$ (see (4.2)). Noting that $\lim_{t\to\infty}\sin(vt)/v=\pi\delta(v)$, the imaginary part of (B 1) is readily obtained as

$$\lim_{t\to\infty}\int_{v_m}^{v_M} h(v)\frac{\sin(vt)}{v}\,\mathrm{d}v=\pi h(0). \tag{B 2}$$

The real part of (B 1) is derived using the equality

$$\int_{v_m}^{v_M} h(v)\frac{1-\cos(vt)}{v}\,\mathrm{d}v=\int_{v_m}^{v_M}\frac{h(v)-h(0)}{v}\,\mathrm{d}v + h(0)\int_{v_m}^{v_M}\frac{1-\cos(vt)}{v}\,\mathrm{d}v-\int_{v_m}^{v_M}\frac{h(v)-h(0)}{v}\cos(vt)\,\mathrm{d}v. \tag{B 3}$$

For large t, the first two terms are $O(1)$ while the third is $O(t^{-1})$ and can therefore be neglected. The second term is calculated according to

$$\int_{\nu_m}^{\nu_M} \frac{1-\cos(\nu t)}{\nu}\,\mathrm{d}\nu = \int_{-\nu_m}^{\nu_M} \frac{1-\cos(\nu t)}{\nu}\,\mathrm{d}\nu$$
$$= \ln\left(\frac{\nu_M}{-\nu_m}\right) - \mathrm{Ci}(\nu_M t) + \mathrm{Ci}(-\nu_m t), \qquad \text{(B 4)}$$

where $\mathrm{Ci}(x)$ is the cosine integral, which vanishes in the limit $x \to +\infty$. Collecting (B 2)–(B 3) yields the final result

$$\lim_{t\to\infty} \int_{\nu_m}^{\nu_M} h(\nu)\left[\frac{1-\cos(\nu t)}{\nu} + \mathrm{i}\frac{\sin(\nu t)}{\nu}\right]\mathrm{d}\nu = \int_{\nu_m}^{\nu_M} \frac{h(\nu)-h(0)}{\nu}\,\mathrm{d}\nu + h(0)\left[\ln\left(\frac{\nu_M}{-\nu_m}\right) + \mathrm{i}\pi\right].$$

Using the definitions of h, ν, ν_m, and ν_M, and introducing the latter expression in (4.7) provides the structure of the streamfunction given by (4.9).

The long-term behaviour of the pseudomomentum P, given by (4.8), is governed by the asymptotic form of

$$\int_0^1 [P_a(y_a)]^{-1}\,|g_a(t;y_a)|^2\,\mathrm{d}y_a \left[\int_0^1 \psi_a(y';y_a)\nabla'^2_a f_{bc}(y')\,\mathrm{d}y'\right]^2,$$

which may be written

$$\int_{\nu_m}^{\nu_M} j(\nu)\frac{\sin^2(\nu t/2)}{(\nu t/2)^2}\,\mathrm{d}\nu, \qquad \text{(B 5)}$$

where

$$j(\nu) := [k_a P_a(y_a)]^{-1}\left[\int_0^1 \psi_a(y';y_a)\nabla'^2_a f_{bc}(y')\,\mathrm{d}y'\right]^2.$$

Examination of (B 5) suggests a linear time-dependence for $t \gg 1$; hence we calculate its approximation in the form

$$t \lim_{t\to\infty} \int_{\nu_m}^{\nu_M} j(\nu)\frac{\sin(\nu t/2)}{(\nu/2)}\frac{\sin(\nu t/2)}{(\nu t/2)}\,\mathrm{d}\nu.$$

Noting that $\lim_{t\to\infty} \sin(\nu t/2)/(\nu/2) = \pi\delta(\nu/2)$, and $\lim_{t\to\infty,\nu\to 0} \sin(\nu t/2)/(\nu t/2) = 1$, we finally obtain

$$\int_{\nu_m}^{\nu_M} j(\nu)\frac{\sin^2(\nu t/2)}{(\nu t/2)^2}\,\mathrm{d}\nu \approx 2\pi t j(0),$$

and hence (4.10).

Appendix C. Calculation of λ in a linear shear

In order to evaluate (2.7) for the particular solution (5.7), we must expand the latter in the vicinity of the critical level. The asymptotic forms of the Whittaker functions are (e.g. Slater 1960)

$$M_{\kappa,1/2}(z_c) = z_c + O(z_c^2),$$

$$W_{\kappa,1/2}(z_c) = \frac{1}{\Gamma(-\kappa)}\left\{-\frac{1}{\kappa} + z_c \ln z_c + z_c\left[\Psi(1-\kappa) - 1 + 2\gamma + \frac{1}{2\kappa}\right]\right\} + O(z_c^2 \ln z_c),$$

where $\Psi(\cdot)$ is the digamma function, and γ is the Euler constant. Introducing these expressions in (5.7), we get the expression for the streamfunction for $y > c$:

$$\psi^{+} = -AM_{\kappa,1/2}(z_{c,0})\left[M_{\kappa,1/2}(z_{c,1})(1-\kappa z_c \ln z_c) + \delta^{+} z_c\right] + O(z_c^2 \ln z_c),$$

where

$$\delta^{+} = \kappa\left\{\Gamma(-\kappa)W_{\kappa,1/2}(z_{c,1}) - M_{\kappa,1/2}(z_{c,1})\left[\Psi(1-\kappa) - 1 + 2\gamma + \frac{1}{2\kappa}\right]\right\}.$$

Similarly, we obtain the streamfunction for $y < c$:

$$\psi^{-} = -AM_{\kappa,1/2}(z_{c,1})\left[M_{\kappa,1/2}(z_{c,0})(1-\kappa z_c \ln(-z_c)) + \delta^{-} z_c\right] + O\left(z_c^2 \ln(-z_c)\right),$$

where

$$\delta^{-} = -\kappa\left\{\Gamma(\kappa)W_{-\kappa,1/2}(-z_{c,0}) + M_{\kappa,1/2}(z_{c,0})\left[\Psi(1+\kappa) - 1 + 2\gamma - \frac{1}{2\kappa}\right]\right\}.$$

With these results, (2.7) becomes

$$\begin{aligned}\lambda &= 2k\left(\frac{\mathrm{d}\psi^{+}}{\mathrm{d}z_c} - \frac{\mathrm{d}\psi^{-}}{\mathrm{d}z_c}\right)_{z_c=0} \\ &= -2kA\left[M_{\kappa,1/2}(z_{c,0})\delta^{+} - M_{\kappa,1/2}(z_{c,1})\delta^{-}\right] \\ &= -A\beta\,\eta,\end{aligned}$$

with

$$\begin{aligned}\eta &= \Gamma(-\kappa)M_{\kappa,1/2}(z_{c,0})W_{\kappa,1/2}(z_{c,1}) + \Gamma(\kappa)M_{\kappa,1/2}(z_{c,1})W_{-\kappa,1/2}(-z_{c,0}) \\ &\quad + M_{\kappa,1/2}(z_{c,0})M_{\kappa,1/2}(z_{c,1})\left[\Psi(1+\kappa) - \Psi(1-\kappa) - \kappa^{-1}\right]. \qquad \text{(C 1)}\end{aligned}$$

REFERENCES

Abarbanel, H. D. I., Holm, D. D., Marsden, J. E. & Ratiu, T. S. 1986 Nonlinear stability analysis of stratified fluid equilibria. *Phil. Trans. R. Soc. Lond.* A **318**, 349–409.

Balmforth, N. J. & Morrison, P. J. 1996 Singular eigenfunctions for shearing fluids. Part 1. *Phys. Fluids*, submitted.

Becker, J. M. & Grimshaw, R. H. J. 1993 Explosive resonant triads in a continuously stratified shear flows. *J. Fluid Mech.* **257**, 219–228.

Benney, D. J. & Maslowe, S. A. 1975 The evolution in space and time of nonlinear waves in parallel shear flows. *Stud. Appl. Maths* **54**, 181–205.

Brown, S. N. & Stewartson K. 1979 On the secular stability of a regular Rossby neutral mode. *Geophys. Astrophys. Fluid Dyn.* **14**, 1–18.

Cairns, R. A. 1979 The role of negative energy waves in some instabilities of parallel flow. *J. Fluid Mech.* **92**, 1–14.

Case, K. M. 1960 Stability of inviscid Couette flow. *Phys. Fluids* **3**, 143–148.

Craik, A. D. D 1985 *Wave Interactions and Fluid Flows*. Cambridge University Press.

Craik, A. D. D. & Adam, J. A. 1979 Explosive resonant interaction in a three-layer fluid flow. *J. Fluid Mech.* **92**, 15–33.

Drazin, P. G., Beaumont, D. N. & Coaker, S. A. 1982 On Rossby waves modified by basic shear, and barotropic instability. *J. Fluid Mech.* **124**, 439–456.

Erdélyi, A., Magnus, W., Oberhettinger, F. & Tricomi, G. F. 1953 *Higher Transcendental Functions*, Vol. I. McGraw Hill.

Farrell, B. F. 1982 The initial growth of a disturbance in a baroclinic flow. *J. Atmos. Sci.* **39**, 1663–1686.

Gakhov, F. D. 1990 *Boundary Value Problems*. Dover.

GRIMSHAW, R. H. J. 1988 Resonant wave interaction in a stratified shear flow. *J. Fluid Mech.* **190**, 357–374.

GRIMSHAW, R. H. J. 1994 Resonant wave interaction near a critical level in a stratified shear flow. *J. Fluid Mech.*, **269**, 1–22.

HELD, I. M. 1985 Pseudomomentum and the orthogonality of modes in shear flows. *J. Atmos. Sci.* **42**, 2280–2288.

KAMP, L. P. J. 1991 Integral-equation approach to the instability of two-dimensional sheared flow of inviscid fluid in a rotating system with variable Coriolis parameter. *J. Phys.* A: *Math. Gen.* **24**, 2029–2052.

KILLWORTH, P. D. & MCINTYRE, M. E. 1985 Do Rossby waves critical layers absorb, reflect or overreflect? *J. Fluid Mech.* **161**, 449–492.

MASLOWE, S. A. 1986 Critical layers in shear flows. *Ann. Rev. Fluid Mech.* **18**, 405–432.

MASLOWE, S. A., BENNEY, D. J. & MAHONEY, D. J. 1994 Wave packet critical layers in shear flows. *Stud. Appl. Maths* **91**, 1–16.

MEACHAM, S. 1988 Non-modal baroclinic instability. *Dyn. Atmos. Oceans* **12**, 19–45.

PEDLOSKY, J. 1987 *Geophysical Fluid Dynamics*. Springer.

RIPA, P. 1981 On the theory of nonlinear wave-wave interactions among geophysical waves. *J. Fluid Mech.* **103**, 87–115.

RIPA, P. 1990 Positive, negative and zero wave energy and the flow stability problem in the Eulerian and Lagrangian-Eulerian descriptions. *Pure Appl. Geophys.* **133**, 713–732.

RITCHIE, H. 1985 Rossby wave resonance in the presence of a nonlinear critical layer. *Geophys. Astrophys. Fluid Dyn.* **31**, 49–92.

ROMANOVA, N. N. 1987 Explosive instability in a three-layered rotating liquid. *Izv. Atmos. Ocean. Phys.* **28**, 343–350.

ROMANOVA, N. N. 1994 Hamiltonian description of weakly nonlinear wave dynamics in non-equilibrium media. *Dokl. Akad. Nauk.* **335**, 592–594 (in Russian).

SHEPHERD, T. G. 1992 Arnol'd stability applied to fluid flow: successes and failures. In *Nonlinear Phenomena in Atmospheric and Oceanic Sciences* (ed. G. F. Carnevale & R. T. Pierrehumbert), pp. 187–206. Springer.

SLATER, L. J. 1960 *Confluent Hypergeometric Functions*. Cambridge University Press.

TRICOMI, F. G. 1985 *Integral Equations*. Dover.

TUNG, K. K. 1983 Initial-value problem for Rossby waves in a shear flow with critical level. *J. Fluid Mech.* **133**, 443–469.

VANNESTE, J. 1995 Explosive resonant interaction of baroclinic Rossby waves and stability of multilayer quasi-geostrophic flow. *J. Fluid Mech.* **291**, 83–107.

VANNESTE, J. & VIAL, F. 1994 On the nonlinear interactions of geophysical waves in shear flows. *Geophys. Astrophys. Fluid Dyn.* **78**, 121–144.

WARN, T. & WARN, H. 1978 The evolution of a nonlinear critical level. *Stud. Appl. Maths* **59**, 37–71.